Analysis questions for teams

1. Suppose that $f \in C_c^\infty(\mathbf{R}^2)$ has the property that $\int_L f = 0$ for every line $L \subset \mathbf{R}^2$. Show that $f = 0$.

2. Let $D \subset \mathbf{C}$ be an open set. Let $Aut(D)$ denote the set of invertible conformal self-maps $\psi : D \to D$.

(a) Show that there is a domain D such that $Aut(D)$ is countably infinite.

(b) Show that there is a domain D such that $Aut(D)$ is isomorphic to the integers $\mathbf{Z}$.

(c) Is it possible for $Aut(D)$ to be isomorphic to the real numbers $\mathbf{R}$?

3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is of class C^1 and satisfies

$$\int_{-\infty}^{\infty} \frac{1}{|f(x)|} dx = +\infty$$

Prove that the maximal interval of existence for the solution of

$$\frac{dx}{dt} = f(x)$$

is $(-\infty, \infty)$.